Univerzitet u Zenici
Filozofski fakultet
Odsjek: Matematika i informatika
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Linearna algebra, pismeni ispit

1. U $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ zadani su potprostori

$$\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a - 2b = 0,\ a + c + d = 0 \right\} \quad \text{i}$$

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + c = 0,\ a - 2b + d = 0 \right\}.$$

Odrediti po jednu bazu za $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M} + \mathcal{N}$ i $\mathcal{M} \cap \mathcal{N}$.

2. Zadana je linearna transformacija $T : \mathcal{P}_3 \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = \begin{bmatrix} a_0 + a_3 - a_2 & a_0 + 2a_1 - a_2 \\ a_3 & a_0 - a_2 \end{bmatrix}$$

Prikažite transformaciju T u paru standardnih baza, te joj odredite $\ker(T)$, $\mathrm{im}(T)$, rang $\rho(T)$ i defekt $\delta(T)$ (rang i defekt linearnog preslikavanja T označavamo redom sa $\rho(T)$ i $\delta(T)$ i definišemo na sljedeći način $\rho(T) := \dim \mathrm{im}(T)$, $\delta(T) := \dim \ker(T)$). ($\mathcal{P}_3$ je prostor svih polinoma stepena ≤ 3).

3. Zadan je linearni operator $T : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ sa

$$T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{pmatrix} a - b & 4a - 4b \\ -a + 2b + c & b + c \end{pmatrix}$$

Odrediti sve jednodimenzionalne potprostore koji su invarijantni u odnosu na operator T.

4. Prostor $\mathcal{L}$ je zadan kao skup rješenja sistema

$$\begin{aligned} 2x_1 + x_2 + x_3 + 3x_4 &= 0 \\ 3x_1 + 2x_2 + 2x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 2x_3 - 9x_4 &= 0\ . \end{aligned}$$

Prikažite vektor $x = (7, -4, -1, 2)^\top$ u obliku $x = y + z$, pri čemu je $y \in \mathcal{L}$, a z iz ortogonalnog komplimenta od $\mathcal{L}$ u $\mathbb{R}^4$.

Važno: Ovaj papir treba predati zajedno s rješenjima zadataka! Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

Zadaci su skinuti sa stranice ff.unze.ba/nabokov.
Za uočene greške pisati na *infoarrt@gmail.com*

#) U $Mat_{2\times 2}(\mathbb{R})$ zadani su podprostori

$$\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a-2b=0,\ a+c+d=0 \right\} \quad i$$

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+c=0,\ a-2b+d=0 \right\}.$$

Odrediti po jednu bazu za $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M}+\mathcal{N}$ i $\mathcal{M}\cap\mathcal{N}$.

$Mat_{2\times 2} \cong \mathbb{R}^4$ kao vektorski prostori

Rj. Primjetimo da matricu $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ možemo tumačiti i kao vektor $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, pa da bi odredili baze za $\mathcal{M}$ i $\mathcal{N}$, puno je jednostavnije posmatrati sljedeća dva vektorska podprostora prostora $\mathbb{R}^4$.

$$\mathcal{M}' = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, a-2b=0,\ a+c+d=0 \right\} \quad i$$

$$\mathcal{N}' = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, a+c=0,\ a-2b+d=0 \right\}.$$

Svedimo sad prostore $\mathcal{M}'$ i $\mathcal{N}'$ na jezgra nekih matrica

$$\mathcal{M}' = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, \underbrace{\begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}}_{=A} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \ker(A)$$

$$\mathcal{N}' = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}}_{=B} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \ker(B)$$

Znamo da kolone iz općeg rješenja sistema $Ax=0$ formiraju bazu za $\ker(A)$.

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right) \overset{II_v - I_v}{\sim} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}\right) \overset{I_v + II_v}{\sim} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}\right) \end{array}$$

$\Rightarrow$ dvije promjenjive uzimamo proizvoljno npr. $c=s$, $d=t$

Rješenja su oblika

$$\begin{aligned} a+c+d&=0 \\ 2b+c+d&=0 \\ \hline a&=-c-d \\ 2b&=-c-d \\ \hline b&=-\tfrac{1}{2}(c+d) \end{aligned}$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -s-t \\ -\frac{1}{2}s-\frac{1}{2}t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ -1/2 \\ 0 \\ 1 \end{pmatrix} t$$

Baza za M je

$$\left\{ \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \right\}$$

Slično, posmatramo $\ker(B)$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \end{array}\right) \overset{II_v - I_v}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 \end{array}\right) \Rightarrow$$

dvije promjenjive uzimamo proizvoljno npr. $c=s$, $d=t$

$$\begin{aligned} a+c&=0 \\ -2b-c+d&=0 \\ \hline a&=-c \\ -2b&=c-d \\ \hline a&=-c \\ b&=-\tfrac{1}{2}c+\tfrac{1}{2}d \end{aligned}$$

Rješenja su oblika

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -s \\ -\frac{1}{2}s+\frac{1}{2}t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} t$$

Baza za N je

$$\left\{ \begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \right\}.$$

Prema definiciji

$$\mathcal{M}+\mathcal{N}=\{m+n \mid m\in\mathcal{M} \text{ i } n\in\mathcal{N}\}$$

Pa da bi našli bazu za $\mathcal{M}+\mathcal{N}$ prvo nađimo bazu za $\mathcal{M}'+\mathcal{N}'$ tj. pronađimo linearno nezavisan skup iz unije baza za $\mathcal{M}'$ i $\mathcal{N}'$. Ili iz definicije od $\mathcal{M}+\mathcal{N}$.

Znamo da

Ako $\mathcal{S}_{\mathcal{X}}, \mathcal{S}_{\mathcal{Y}}$ generišu $\mathcal{X}, \mathcal{Y}$ tada $\mathcal{S}_{\mathcal{X}} \cup \mathcal{S}_{\mathcal{Y}}$ generiše $\mathcal{X}+\mathcal{Y}$.

Primjetimo da

$$\mathcal{M}'=\ker(A)=\operatorname{span}\left\{\begin{pmatrix}-1\\-\frac{1}{2}\\1\\0\end{pmatrix},\begin{pmatrix}-1\\+\frac{1}{2}\\0\\1\end{pmatrix}\right\}=\operatorname{im}\left(\begin{bmatrix}-1&-1\\-\frac{1}{2}&-\frac{1}{2}\\1&0\\0&1\end{bmatrix}\right)$$

$$\mathcal{N}'=\ker(B)=\operatorname{span}\left\{\begin{pmatrix}-1\\-\frac{1}{2}\\1\\0\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\0\\1\end{pmatrix}\right\}=\operatorname{im}\left(\begin{bmatrix}-1&0\\-\frac{1}{2}&\frac{1}{2}\\1&0\\0&1\end{bmatrix}\right)$$

Znamo $\operatorname{im}(A^T)=\operatorname{im}(B^T)$ akko $A \overset{\text{red}}{\sim} B$

$$\begin{pmatrix}-1&-\frac{1}{2}&1&0\\-1&-\frac{1}{2}&0&1\\-1&-\frac{1}{2}&1&0\\0&\frac{1}{2}&0&1\end{pmatrix} \underset{III_v-I_v}{\overset{II_v-I_v}{\sim}} \begin{pmatrix}-1&-\frac{1}{2}&1&0\\0&0&-1&1\\0&0&0&0\\0&\frac{1}{2}&0&1\end{pmatrix} \overset{II_v\leftrightarrow IV_v}{\sim} \begin{pmatrix}-1&-\frac{1}{2}&1&0\\0&\frac{1}{2}&0&1\\0&0&0&0\\0&0&-1&1\end{pmatrix}$$

Baza za $\mathcal{M}+\mathcal{N}$ je

$$\left\{\begin{pmatrix}-1&-\frac{1}{2}\\1&0\end{pmatrix},\begin{pmatrix}0&\frac{1}{2}\\0&1\end{pmatrix},\begin{pmatrix}0&0\\-1&1\end{pmatrix}\right\}$$

$$M' \cap N' = \{ x \in \mathbb{R}^4 \mid x \in M' \text{ i } x \in N' \}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; a-2b=0,\ a+c+d=0,\ a+c=0,\ a-2b+d=0 \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \underbrace{\begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}}_{=C} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \ker(C)$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \end{array}\right) \underset{IV_v - I_v}{\overset{\substack{II_v - I_v \\ III_v - I_v}}{\sim}} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \overset{III_v - II_v}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \underset{II_v + III_v}{\overset{IV_v + III_v}{\sim}} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

1 promenjivu uzimamo proizvoljno

$a-2b=0$

$2b+c=0$

$-d=0$

$a=2b$

$2b=-c \Rightarrow b=-\frac{1}{2}s$

$a=-s$

$c=s$

$s \in \mathbb{R}$

rješenje je

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -s \\ -\frac{1}{2}s \\ s \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} s$$

Baza za $M \cap N$ je $\left\{ \begin{pmatrix} -1 & -1/2 \\ 1 & 0 \end{pmatrix} \right\}$.

(#) Zadan je linearni operator $T: \mathcal{P}_3 \to Mat_{2\times 2}(\mathbb{R})$:

$$T(a_0+a_1t+a_2t^2+a_3t^3)=\begin{pmatrix} a_0+a_3-a_2 & a_0+2a_1-a_2 \\ a_3 & a_0-a_2 \end{pmatrix}.$$

Prikažite operator T u paru standardnih baza, te mu odredite $\ker(T)$, $\operatorname{im}(T)$, rang $\rho(T)$ i defekt $\delta(T)$ (rang i defekt linearnog preslikavanja T označavamo redom sa $\rho(T)$ i $\delta(T)$ i definišemo sa $\rho(T)=\dim \operatorname{im}(T)$, $\delta(T)=\dim\ker(T)$). ($\mathcal{P}_3$ je prostor polinoma stepena ≤ 3).

Rj. Standardna baza za $\mathcal{P}_3$ je $\{1, t, t^2, t^3\}$, a standardna baza za $Mat_{2\times2}$ je $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$.

Prisjetimo se:

Ako su $\mathcal{B}=\{u_1,u_2,\dots,u_n\}$ i $\mathcal{B}'=\{v_1,v_2,\dots,v_n\}$ redom baze za $\mathcal{U}$ i $\mathcal{V}$ tada

$$[T]_{\mathcal{B}\mathcal{B}'}=\begin{pmatrix} | & | & & | \\ [T(u_1)]_{\mathcal{B}'} & [T(u_2)]_{\mathcal{B}'} & \dots & [T(u_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}$$

Kako je $T(1)=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $T(t)=\begin{pmatrix}0&2\\0&0\end{pmatrix}$, $T(t^2)=\begin{pmatrix}-1&-1\\0&-1\end{pmatrix}$, $T(t^3)=\begin{pmatrix}1&0\\1&0\end{pmatrix}$ to je

$$[T]_{\mathcal{S}\mathcal{S}'}=\begin{pmatrix} 1&0&-1&1\\ 1&2&-1&0\\ 0&0&0&1\\ 1&0&-1&0 \end{pmatrix}$$

operator T u paru standardnih baza

gdje su $\mathcal{S}=\{1,t,t^2,t^3\}$, $\mathcal{S}'=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$,

Prema definicije

$$\ker T=\{p\in\mathcal{P}_3 \mid T(p)=0\}.$$

Ako ovo drugačije napišemo imamo

$$\ker T = \{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_0 + a_3 - a_2 = 0,\ a_0 + 2a_1 - a_2 = 0,\ a_3 = 0,\ a_0 - a_2 = 0 \}$$

$$= \left\{ a_0 + a_1 t + a_2 t^2 + a_3 t^2 \;\middle|\; \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{array}\right) \sim \ldots \sim \begin{array}{c} \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \Rightarrow$$ jednu promjenjivu uzimamo proizvoljno npr. $a_2 = s$.

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ s \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} s, \quad s \in \mathbb{R}$$

$\ker(T) = \{ a + at^2 \mid a \in \mathbb{R} \}$ jezgro operatora T $\Rightarrow \dim \ker(T) = 1$

$\Rightarrow \rho(T) = 1$

Prema definiciji $\operatorname{im}(T) = \{ T(p) \mid p \in \mathcal{P}_3 \} = \{ T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$

$$= \left\{ \begin{pmatrix} a_0 + a_3 - a_2 & a_0 + 2a_1 - a_2 \\ a_3 & a_0 - a_2 \end{pmatrix} \;\middle|\; a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}$$ pa posmatrajmo skup

$$\left\{ \begin{pmatrix} a_0 + a_3 - a_2 \\ a_0 + 2a_1 - a_2 \\ a_3 \\ a_0 - a_2 \end{pmatrix} \;\middle|\; a_0, a_1, a_2, a_3 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \;\middle|\; a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}$$

Prisjetimo se da <u>osnovne kolone u A generišu $\operatorname{im}(A)$</u>.

Prema tome

$$\operatorname{im}(T) = \operatorname{span}\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\} \Rightarrow \delta(T) = 3.$$

(#) Zadan je linearni operator $T: Mat_{2\times2}(\mathbb{R}) \to Mat_{2\times2}(\mathbb{R})$ sa

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{pmatrix} a-b & 4a-4b \\ -a+2b+c & b+c \end{pmatrix}.$$

Odrediti sve jednodimenzionalne potprostore koji su invarijantni u odnosu na operator T.

Rj: Prisjetimo se

<u>Invarijantni potprostor</u>
<u>Za potprostor $\mathcal{X} \subseteq V$ kažemo da je invarijantan potprostor u odnosu na T (T je linearni operator na V) akko</u>

$$T(\mathcal{X}) \subseteq \mathcal{X}$$

Mi trebamo odrediti (pronaći) potprostor $\mathcal{M}$ dimenzije 1 koji je invarijantan u odnosu na T.

Neka je $\mathcal{M} = \text{span}\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right\}$ gdje je $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ tražena baza.

$$T(\mathcal{M}) \subseteq \mathcal{M} \overset{\mathcal{M}=\text{span}\{A\}}{\Longrightarrow} \forall B \in \mathcal{M} \quad T(B) = \alpha \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

gdje je $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ baza za $\mathcal{M}$. Ali kako je $B \in \mathcal{M}$ to se B može prikazati kao linearna kombinacija matrica iz $\underbrace{\ }_{\text{baze od}}$ $\mathcal{M}$ tj. $\exists \beta \in \mathbb{R}$ t.d. $B = \beta \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. Prema tome

$$T(B) = \alpha \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \Rightarrow \beta\, T\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = \alpha \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_1 - a_2 & 4a_1 - 4a_2 \\ -a_1 + 2a_2 + a_3 & a_2 + a_3 \end{bmatrix} = \frac{\alpha}{\beta} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \overset{\lambda = \frac{\alpha}{\beta}}{\Longrightarrow}$$

$$
\begin{aligned}
a_1 - a_2 &= \lambda a_1 \\
4a_1 - 4a_2 &= \lambda a_2 \\
-a_1 + 2a_2 + a_3 &= \lambda a_3 \\
a_2 + a_3 &= \lambda a_4
\end{aligned}
\quad \Rightarrow \quad
\underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{C}
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}
$$

$$\Downarrow$$

$$Ca = \lambda a \quad \Rightarrow \quad (C - \lambda I)a = 0$$
$$a \neq 0$$

$$
\begin{vmatrix} 1-\lambda & -1 & 0 & 0 \\ 4 & -4-\lambda & 0 & 0 \\ -1 & 2 & 1-\lambda & 0 \\ 0 & 1 & 1 & -\lambda \end{vmatrix} = \ldots = \lambda^2(\lambda - 1)\cdot(\lambda + 3)
$$

$\lambda_1 = 0$:

$$
\overline{C} = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right] \sim \ldots \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xRightarrow[\text{upr } a_3 = s,\ a_4 = t,\ s,t \in \mathbb{R}]{\text{2 pr. uz pr.}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -s \\ -s \\ s \\ t \end{pmatrix}, s,t \in \mathbb{R}
$$

$\lambda_2 = 1$:

$$
\overline{C} = \left[\begin{array}{cccc|c} 0 & -1 & 0 & 0 & 0 \\ 4 & -5 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right] \sim \ldots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xRightarrow[\text{upr } x_4 = s]{\text{1 pr. uz pr.}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ s \\ s \end{pmatrix}, s \in \mathbb{R}
$$

$\lambda_3 = -3$:

$$
\overline{C} = \left[\begin{array}{cccc|c} 4 & -1 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 & 0 \\ -1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 \end{array}\right] \sim \ldots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & 16/3 & 0 \\ 0 & 0 & 1 & -7/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xRightarrow[\text{upr } x_4 = 3s]{\text{1 pr. uz pr.}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -4s \\ -16s \\ 7s \\ 3s \end{pmatrix}, s \in \mathbb{R}
$$

Jednodimenzionalni potprostori koji su invarijantni u odnosu na operator T su

$$
M_1 = \operatorname{span}\left\{ \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \quad M_2 = \operatorname{span}\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\} \quad \text{i}
$$

$$
M_3 = \operatorname{span}\left\{ \begin{bmatrix} -4 & -16 \\ 7 & 3 \end{bmatrix} \right\}.
$$

Prostor $\mathcal{L}$ je zadan kao skup rješenja sistema

$$\begin{aligned} 2x_1 + x_2 + x_3 + 3x_4 &= 0 \\ 3x_1 + 2x_2 + 2x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 2x_3 - 9x_4 &= 0 \end{aligned}$$

Prikažite vektor $x = (7, -4, -1, 2)$ u obliku $x = y + z$, pri čemu je $y \in \mathcal{L}$, a z iz ortogonalnog komplementa od $\mathcal{L}$ u $\mathbb{R}^4$.

Rj. Riješimo dati sistem

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 3 & 0 \\ 3 & 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & -9 & 0 \end{array}\right) \sim \dots \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \Rightarrow$$

dvije promj. u žin. proizv. npr. $x_3 = s$, $x_4 = t$.

Rješenje sistema je

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5t \\ -s+7t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} s, \quad s, t \in \mathbb{R}$$

Prema tome $\mathcal{L} = \operatorname{span}\left\{ \begin{pmatrix} -5 \\ 7 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Prisjetimo se

Za podskup M unitarnog prostora V ortogonalni komplement od M definišemo sa

$$M^{\perp} = \{ x \in V \mid \langle m, x \rangle = 0 \text{ za } \forall m \in M \}$$

Iz definicije vidimo da trebamo odrediti vektore $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ za koje vrijedi

$$\begin{aligned} -5x_1 + 7x_2 + x_4 &= 0 \\ -x_2 + x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} -5 & 7 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right] \sim \ldots \sim \left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{5} & -\frac{1}{5} & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

dvije promjenjive uzimu proizv.
npr. $x_3 = 5s$
$x_4 = 5t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7s+t \\ 5s \\ 5s \\ 5t \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 5 \\ 0 \end{pmatrix} s + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 5 \end{pmatrix} t$$

Prema tome
$$\mathcal{L}^{\perp} = \operatorname{span}\left\{ \begin{pmatrix} 7 \\ 5 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 5 \end{pmatrix} \right\}.$$

Ostalo je još da odredimo $\alpha, \beta, \gamma, \delta$ za koje vrijedi

$$\begin{pmatrix} 7 \\ -4 \\ -1 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} -5 \\ 7 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 7 \\ 5 \\ 5 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

⋮ za vježbu

$$\alpha = -1, \quad \beta = -2, \quad \gamma = \frac{1}{5}, \quad \delta = \frac{3}{5}$$

$$Y = (-1)\begin{pmatrix} -5 \\ 7 \\ 0 \\ 1 \end{pmatrix} + (-2)\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ -2 \\ -1 \end{pmatrix} \in \mathcal{L}$$

$$Z = \frac{1}{5}\begin{pmatrix} 7 \\ 5 \\ 5 \\ 0 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} \in \mathcal{L}^{\top}$$

Prema tome
$$\underbrace{\begin{pmatrix} 7 \\ -4 \\ -1 \\ 2 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 5 \\ -5 \\ -2 \\ -1 \end{pmatrix}}_{Y} + \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}}_{Z}$$

traženo rješenje